Solution of Quasi-Linear Parabolic Equations with Discontinuous Coefficients (ポテンシャル論と偏微分方程式)

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§1. Introduction.

This paper is concerned with weak solutions of a quasi-linear parabolic differential equation

$$(1.1)\qquad Lu = u_t - \operatorname{div} \mathcal{A}(x, t, u, u_x) + B(x, t, u, u_x) = 0$$

under the conditions

$$\begin{cases} |\mathcal{A}(x, t, u, p)| \leqq \bar{a}|p|^{\alpha-1} + c(x, t)|u|^{\alpha-1} + e(x, t) \\ |B(x, t, u, p)| \leqq b(x, t)|p|^{\alpha-1} + d(x, t)|u|^{\alpha-1} + f(x, t) \\ p\cdot\mathcal{A}(x, t, u, p) \geqq a|p|^{\alpha} - d(x, t)|u|^{\alpha} - g(x, t), \end{cases}$$

where $\alpha > 2$, p is any n-dimensional real vector, a and $\bar{a}$ are positive constants, $b, c, \cdots, g$ are positive functions of (x, t) belonging to some Lebesgue classes. $\mathcal{A}(x, t, u, u_x)$ is a given vector function of (x, t, u, u_x), $B(x, t, u, u_x$ a given scalar function of the same variables, and $u_x = (\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$. Also div $\mathcal{A}$ in (1.1) in refers to the divergence of the variables $(x_1, \cdots, x_n)$.

Our aim is to derive the local behavior of solutions ; local boundedness and the Hölder continuity of solutions.

In 1964, J. Moser [3] proved the boundedness of solutions and the Harnack inequality for the linear parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x, t)\frac{\partial u}{\partial x_j}\right),$$

where $a_{ij} = a_{ji}$ are bounded and measurable and

$$\lambda|\xi|^2 \leq \Sigma a_{ij}\xi_i\xi_j \leq M|\xi|^2 \quad \text{for some constants } \lambda, M.$$

J. Serrin [4] proved these results for the quasi-linear elliptic equation

$$-\operatorname{div} \mathcal{A}(x, u, u_x) + B(x, u, u_x) = 0$$

under the condition

$$\begin{cases} |\mathcal{A}(x, u, p)| < a|p|^{\alpha-1} + c(x)|u|^{\alpha-1} + e(x), \\ |B(x, u, p)| < b(x)|p|^{\alpha-1} + d(x)|u|^{\alpha-1} + f(x), \\ p\cdot\mathcal{A}(x, u, p) > |p|^{\alpha} - d(x)|u|^{\alpha} - g(x), \end{cases}$$

where a is a positive constant and if $1 < \alpha < n$,

$$c, e \in L_{n/(\alpha-1)}, \quad b \in L_{n/(1-\varepsilon)}, \quad d, f, g \in L_{n/(\alpha-\varepsilon)},$$

and if $\alpha = n$

$$c, e \in L_{n/(n-1)+\varepsilon}, \quad b \in L_{n+\varepsilon}, \quad d, f, g \in L_{1+\varepsilon}$$

and if $\alpha > n$

$$c, e \in L_{\alpha/(\alpha-1)}, \quad b \in L_{\alpha}, \quad d, f, g \in L_1.$$

These results are extended by J. Serrin and D. G. Aronson [1] to the quasi-linear parabolic equation

$$u_t = \operatorname{div} \mathcal{A}(x, t, u, u_x) + B(x, t, u, u_x),$$

where

$$\begin{cases} p \cdot \mathcal{A}(x, t, u, p) \geq a|p|^2 - b^2|u|^2 - f^2, \\ B(x, t, u, p) \leq c|p| + d|u| + g, \\ \mathcal{A}(x, t, u, p) \leq \bar{a}|p| + e|u| + h. \end{cases}$$

Here a and $\bar{a}$ are positive constants, while the coefficients $b, c, \cdots, h$ are non-negative functions of (x, t) and each coefficient is contained in some space $L^{p,q}(\Omega)$, where

$$p > 2 \quad \text{and} \quad \frac{n}{2p} + \frac{1}{q} < \frac{1}{2} \quad \text{for} \quad b, c, e, f, h,$$

and

$$p > 1 \quad \text{and} \quad \frac{n}{2p} + \frac{1}{q} < 1 \quad \text{for} \quad d, g.$$

Also J. Serrin and D. G. Aronson [1] proved the maximum principle for the non-linear parabolic equation (1. 1) under the conditions

$$\begin{cases} p \cdot \mathcal{A}(x, t, u, p) \geq a|p|^{\alpha} - b^{\alpha}|u|^{\alpha} - f^{\alpha}, \\ |B(x, t, u, p)| \leq c|p|^{\alpha-1} + d^{\alpha-1}|u|^{\alpha-1} + g^{\alpha-1}, \end{cases}$$

where $a, b, \cdots, g$ are non-negative constants.

By similar methods as in [2], we can easily obtain the maximum principle for the equation (1. 1) even if the coefficients $b, c, \cdots, g$ are measurable functions of (x, t).

We shall prove boundedness of solutions in § 3, § 4 and § 5 are devoted to prove the Hölder continuity of solutions.

§ 2. Fundamental inequalities.

Let $x = (x_1, \cdots, x_n)$ denote points in n-dimensional Euclidean space E^n $(n > 1)$ and t denote points on the real line. Let Ω be a bounded domain in E^n, and consider the space-time cylinder $Q = \Omega \times (0, T)$ for some fixed $T > 0$. For $(x, t) \in Q$ we consider the quasilinear parabolic equation

$$Lu = u_t - \operatorname{div} \mathcal{A}(x, t, u, u_x) + B(x, t, u, u_x), \tag{2.1}$$

where $\mathcal{A}(x, t, u, u_x)$ is a given vector function of (x, t, u, u_x) and $B(x, t, u, u_x)$ a given scalar function of the same variables.

Throughout this paper we shall assume that $\mathcal{A}(x, t, u, p)$ and $B(x, t, u, p)$ are defined and measurable for all $(x, t) \in Q$ and for all values of u and p, and satisfy inequalities of the form

$$\begin{cases} |\mathcal{A}(x, t, u, p)| \leqq \bar{a}|p|^{\alpha-1} + c(x, t)|u|^{\alpha-1} + e(x, t), \\ |B(x, t, u, p)| \leqq b(x, t)|p|^{\alpha-1} + d(x, t)|u|^{\alpha-1} + f(x, t), \\ p \cdot \mathcal{A}(x, t, u, p) \geqq a|p|^{\alpha} - d(x, t)|u|^{\alpha} - g(x, t) \end{cases} \tag{2.2}$$

for any n-dimensional real vector p and $\alpha > 2$, where a and $\bar{a}$ are positive constants. Moreover, we assume that

$$
(2.\ 3)\quad \begin{cases} \text{if}\quad a < n \\ b \in L^{\infty}[0,\ T; L^{n+\varepsilon}(\Omega)],\ \ c,\ e \in L^{\infty}[0,\ T; L^{n/(a-1)}(\Omega)],\ \ d,\ f,\ g \in L^{\infty}[0,\ T; L^{\frac{n}{a}+\varepsilon}(\Omega)] \\ \text{and if}\quad n \leqq a \\ b \in L^{\infty}[0,\ T; L^{a(1+2\varepsilon)}(\Omega)],\ \ c,\ e \in L^{\infty}[0,\ T\ ;\ L^{\frac{a}{a-1}(1+\varepsilon)}(\Omega), \\ d,\ f,\ g \in L^{\infty}[0,\ T\ ;\ L^{1+2\varepsilon}(\Omega)]\quad \text{for a sufficiently small}\quad \varepsilon > 0, \end{cases}
$$

and that

$$
(2.\ 4)\qquad \|b\| + \|c\| + \|d\| + \|e\| + \|f\| + \|g\| \leqq M
$$

for a positive constant M, where $\|\cdot\|$ denotes the norm in respective space.

Let $L^q[0,\ T\ ;\ L^p(\Omega)]$ be the space of functions $\varphi(x,\ t)$ with the following properties :

(i) φ is defined and measurable in $Q = \Omega \times (0,\ T)$,

(ii) for almost all $t \in (0,\ T)$, $\varphi(x,\ t) \in L^p(\Omega)$,

(iii) $\|\varphi\|_{L^p(\Omega)}(t) \in L^q(0,\ T)$.

Here $\|\varphi\|_{L^p(\Omega)}(t) = (\int_{\Omega} \varphi(x,\ t)^p dx)^{\frac{1}{p}}$.

In the following sections we shall derive various bounds for weak solutions of the equation (2. 1). The material of the present section is basic for this purpose.

We say that u is a weak solution of the equation (2. 1) in Q if $u \in L^{\infty}[0,\ T\ ;\ L^a(\Omega)] \cap L^a[0,\ T; H^{1,a}(\Omega)]$ and satisfies the following

$$(2.\ 5)\qquad \iint_Q \{-u\phi_t + \phi_x \mathcal{A}(x,\ t,\ u,\ u_x) + \phi \cdot B(x,\ t,\ u,\ u_x)\}dxdt = 0$$

for any $\phi \in C_0^1(Q)$.

First we consider a less general class of weak solutions, namely $u \in H^{1,\alpha}[0, T ; L^\alpha(\Omega)] \cap L^\alpha[0, T ; H^{1,\alpha}(\Omega)]$. The weak form (2. 5) of the equation (2. 1) can then be replaced by

$$(2.\ 6)\qquad \iint_Q \{\phi u_t + \phi_x \cdot \mathcal{A}(x,\ t,\ u,\ u_x) + \phi \cdot B(x,\ t,\ u,\ u_x)\}dxdt = 0$$

for any $\phi \in C_0^1(Q)$.

It is easy to see that (2. 6) is not only valid for any functions in $C_0^1(Q)$ but also for any functions in $L^\alpha[0, T ; H^{1,\alpha}(\Omega)]$ with compact support in Q.

Now we derive some inequalities which are required in the following sections.

Put $\bar{u} = \max(0, u) + k$ for some constant $k > 0$ and define the function

$$F(u) = \begin{cases} \bar{u}^\beta - k^\beta & \text{for } -\infty < u \leqq \ell - k \\ \ell^{\beta-1}\bar{u} - k^\beta & \text{for } \ell - k \leqq u < \infty, \end{cases}$$

where $\beta = \alpha q - \alpha + 1$ with any $q > 1$ and ℓ is an arbitrary number larger than k.

Let $\varphi = \varphi(x, t)$ be a piecewise smooth non-negative function vanishing in a neighborhood of the parabolic boundary set $\Gamma = \{\partial\Omega \times (0, T)\} \cup \{\Omega \times (t = 0)\}$ and let $\chi(t ; \tau)$ be the characteristic function of the open interval $(0, \tau)$, $(0 < \tau < T)$. We set $\phi(x, t) = \varphi^\alpha F(u)\chi(t ; \tau)$. Since $\varphi^\alpha\chi(t ; \tau)$ has compact support in Q and $F(u) \in L^\alpha[0, T ; H^{1,\alpha}(\Omega)]$, the function $\phi(x, t)$ may be a test function. Thus we have

(2. 7) $$\iint_Q [\phi u_t + \phi_x \cdot \mathcal{A}(x, t, u, u_x) + \phi \cdot B(x, t, u, u_x)]\,dxdt = 0.$$

On the set where $\phi > 0$ we have necessarily $u > 0$, and consequently $u_x = \overline{u}_x$ and $u < \overline{u}$. Therefore on this set, using the condition (2. 2)

$$\begin{aligned}
&\phi u_t + \phi_x \cdot \mathcal{A}(x, t, u, u_x) + \phi \cdot B(x, t, u, u_x) \\
&= \phi u_t + (\varphi^a \overline{u}_x F' + a\varphi_x \varphi^{a-1} F) \cdot \mathcal{A} + \varphi^a F B \\
&\geqq \phi \overline{u}_t + \varphi^a F' \{a|\overline{u}_x|^a - d|\overline{u}|^a - g\} - a|\varphi_x|\varphi^{a-1} F\{\overline{a}|\overline{u}_x|^{a-1} + c|\overline{u}|^{a-1} \\
&\quad + e\} - \varphi^a F\{b|\overline{u}_x|^{a-1} + d|\overline{u}|^{a-1} + f\}.
\end{aligned}$$

Put

$$H(u) = \begin{cases} \dfrac{1}{\beta+1}\{\overline{u}^{\beta+1} - (\beta+1)k^\beta \overline{u} + \beta k^{\beta+1}\} & \text{for } -\infty < u \leqq \ell - k \\[2ex] \dfrac{1}{2}\ell^{\beta+1}\overline{u}^2 - k^\beta \overline{u} + \dfrac{1}{\beta+1}(\beta k^{\beta+1} - \dfrac{\beta-1}{2}\ell^{\beta+1}) & \text{for } \ell - k < u < \infty. \end{cases}$$

Then $H'(u) = F(u)$. Hence we see

$$\phi \overline{u}_t = \varphi^a F(u)\overline{u}_t = (\varphi^a H)_t - a\varphi^{a-1}\varphi_t H(u).$$

Since $F \leqq \overline{u}^\beta$ and $k \leqq \overline{u}$, we have

(2. 8) $$\begin{aligned}
&(\varphi^a H)_t + a\varphi^a F'(u)\,\overline{u}_x{}^a \leqq \phi u_t + \phi_x \cdot \mathcal{A} + \phi \cdot B + (a\overline{a}|\varphi_x|\varphi^{a-1} \\
&\quad + b\varphi^a)\overline{u}^{\beta+1}|\overline{u}_x|^{a-1} + G_0(x, t)\overline{u}^{\beta+a-1} + \varphi^a F'(d + k^{-a}g)\overline{u}^a + a\varphi^{a-1}|\varphi_t|H(u),
\end{aligned}$$

where $G_0(x, t) = ac|\varphi_x|\varphi^{a-1} + d\varphi^a + ak^{1-a}e|\varphi_x|\varphi^{a-1} + k^{1-a}f\varphi^a$.

The inequality (2. 8) has been proved to hold only in the set where $\phi > 0$. When $\phi = 0$ and $t < \tau$, we have either $\varphi = 0$ or $u < 0$ and $\overline{u}_x = 0$. In either case, (2. 8) clearly remains valid ; thus in fact (2. 8) holds throughout

$\Omega\times(0,\ T)$. Integrating (2. 8) over $\Omega\times(0,\ T)$, we have

$$\int_{\Omega}\varphi^{a}H]_{t=\tau}dx + a\iint\varphi^{a}F'|\overline{u}_{x}|^{a}dxdt$$
$$\leqq \iint[(a\bar{a}|\varphi_{x}|\varphi^{a-1} + b\varphi^{a})\overline{u}^{\beta+1}|\overline{u}_{x}|^{a-1} + G_{0}(x,\ t)\overline{u}^{\beta+a-1}$$
$$+ \varphi^{a}F'(d + k^{-a}g)\overline{u}^{a}]dxdt + a\iint\varphi^{a-1}|\varphi_{t}|Hdxdt.$$

At this stage it is possible to let the parameter ℓ tend to infinity. Since F' and H are non-decreasing functions of ℓ, that is

$$\lim_{\ell\to\infty} H = \frac{1}{\beta+1}\{\overline{u}^{\beta+1} - (\beta+1)k^{\beta}\overline{u} + \beta k^{\beta+1}\} \leqq \frac{1}{\beta+1}\overline{u}^{\beta+1}$$

$$\lim_{\ell\to\infty} F' = \beta\overline{u}^{\beta-1},$$

by applying the monotone convergence theorem, we obtain for almost all values τ in $(0,\ T)$

$$(2.\ 9)\qquad \frac{1}{\beta+1}\int_{\Omega}\varphi^{a}\{\overline{u}^{\beta+1} - (\beta+1)k^{\beta}\overline{u} + \beta k^{\beta+1}\}_{t=\tau}dx + a\beta\iint\varphi^{a}\overline{u}^{\beta-1}|\overline{u}_{x}|^{a}dxdt$$
$$\leqq \iint[(a\bar{a}|\varphi_{x}|\varphi^{a-1} + b\varphi^{a})\overline{u}^{\beta+1}|\overline{u}_{x}|^{a-1} + G_{1}(x,\ t)\overline{u}^{\beta+a-1}]dxdt$$
$$+ \frac{a}{\beta+1}\iint\varphi^{a-1}|\varphi_{t}|\overline{u}^{\beta+1}dxdt,$$

where $G_{1}(x,\ t) = G_{0}(x,\ t) + \beta d\varphi^{a} + \beta k^{-a}g\varphi^{a}$.

The inequality (2. 9) will be used in § 3. In § 4 and § 5 another variant of (2. 9) will be required.

Suppose that u is non-negative and locally bounded in Ω. We put $\overline{u} = u + k$ and define $F(u) = \overline{u}^{\beta}$, where $\beta = aq - a + 1$ for $q \geqq 1$ or $q < \frac{a-2}{a}$ $(q = 0)$.

We take $\phi(x,\ t) = \varphi^{a}(x,\ t)F(u)\chi(t\ ;\ \tau)$ as a test function, where φ and χ are as before.

By a similar calculation we have

$$(2.\ 10)\qquad \frac{1}{|\beta+1|}\Big[\int \varphi^{a}\bar{u}^{\beta+1}dx\Big]_{t=\tau} + a\beta\iint \varphi^{a}\bar{u}^{\beta-1}|\bar{u}_x|^{a}\,dxdt$$

$$\leqq \iint [(a\bar{a}|\varphi_x|\varphi^{a-1} + b\varphi^{a})\bar{u}^{\beta}|\bar{u}_x|^{a-1} + G_2(x,\ t)\bar{u}^{\beta+a-1}]dxdt$$

$$+\frac{a}{|\beta+1|}\iint \varphi^{a-1}|\varphi_t|\bar{u}^{\beta+1}dxdt$$

if $q \geqq 1$, that is $\beta > 0$, where $G_2(x,\ t) = G_0(x,\ t) + |\beta|d\varphi^{a} + |\beta|k^{-a}g\varphi^{a}$.

If $q < \frac{a-2}{a}$, then $\beta+1 < 0$. Of course $\beta < 0$. So we have

$$\phi u_t + \phi_x\cdot A + \phi\cdot B = \varphi^{a}\bar{u}^{\beta}\bar{u}_t + (\beta\bar{u}_x\bar{u}^{\beta-1}\varphi^{a} + a\varphi_x\varphi^{a-1}\bar{u}^{\beta})\cdot A + \varphi^{a}\bar{u}^{\beta}B$$

$$\leqq \frac{1}{|\beta+1|}(\varphi^{a}\bar{u}^{\beta+1})_t - \frac{a}{\beta+1}|\varphi_t|\varphi^{a-1}\bar{u}^{\beta+1} - \beta\bar{u}^{\beta-1}\varphi^{a}(a|\bar{u}_x|^{a} - d\,\bar{u}^{a} - g)$$

$$+ a|\varphi_x|\varphi^{a-1}\bar{u}^{\beta}(\bar{a}|\bar{u}_x|^{a-1} + c\,\bar{u}^{\,a-1} + e) + \varphi^{a}\bar{u}^{\beta}(b|\bar{u}_x|^{a-1} + d\bar{u}^{a-1} + f).$$

Integrating this inequality over $\Omega\times(0,\ \tau)$, we see that (2. 10) is also valid when $q < \frac{a-2}{a}$.

Remark. The inequalities (2. 9) and (2. 10) are also valid for a weak solution $u \in L^{\infty}[0,\ T\ ;\ L^{a}(\Omega)] \cap L^{a}[0,\ T\ ;\ H^{1,a}(\Omega)]$ of the equation (2. 1). (cf. [1]).

At the end of this section, we state three lemmas which will be often used in the latter.

Lemma 2. 1 (cf. [4]). If $\psi(x) \in H_0^{1,a}(\Omega)$ for $a < n$, then

$$\|\psi\|_{a^*} \leqq S\|\psi_x\|_{a}\qquad \Big(\frac{1}{a^*} = \frac{1}{a} - \frac{1}{n}\Big),$$

where the constant S depends only on a and n.

Lemma 2. 2. If $\psi(x) \in H_0^{1,a}$ for $(a \geqq n)$, then for any $\varepsilon > 0$,

$$\|\psi\|_{\frac{a(1+\varepsilon)}{\varepsilon}} \leqq S\|\psi\|_a |\Omega|^{\frac{a(1+\varepsilon)-n}{na(1+\varepsilon)}} .$$

Proof. Since $\frac{a(1+\varepsilon)}{\varepsilon} = \{\frac{na(1+\varepsilon)}{n\varepsilon + a(1+\varepsilon)}\}^*$, using Lemma 2.1, we see

$$\left(\int_\Omega |\psi|^{\frac{a(1+\varepsilon)}{\varepsilon}} dx\right)^{\frac{\varepsilon}{a(1+\varepsilon)}} = \left(\int_\Omega |\psi|^{\{\frac{na(1+\varepsilon)}{n\varepsilon+a(1+\varepsilon)}\}^*} dx\right)^{\frac{\varepsilon}{a(1+\varepsilon)}}$$

$$\leqq S\left(\int_\Omega |\psi_x|^{\frac{na(1+\varepsilon)}{n\varepsilon+a(1+\varepsilon)}} dx\right)^{\frac{n\varepsilon+a(1+\varepsilon)}{na(1+\varepsilon)}} \leqq S\left(\int_\Omega |\psi_x|^a dx\right)^{\frac{1}{a}} |\Omega|^{\frac{a(1+\varepsilon)-n}{na(1+\varepsilon)}} .$$

Lemma 2. 3. If $f(x, t)$ belongs to $L^\infty[0, T\,; L^q(\Omega)]$ and if $\max_{0<t<T}\left(\int_\Omega |f|^q dx\right)^{\frac{1}{q}}$ $\leqq M$ for $q > p > 1$, then $f(x, t)$ can be written in the form $f(x, t) = f'(x, t) + f''(x, t)$, where $\max_t \left(\int_\Omega |f''|^p dx\right)^{\frac{1}{p}} \leq \eta$ and $\sup_Q |f'| \leqq K(\eta)$ for any $\eta > 0$ and for a positive function $K(\eta)$ of η. Moreover $K(\eta)$ may be taken as the value $c(M)\eta^{-\frac{p}{q-p}}$, where $c(M)$ is a constant depending only on M, p and q. (cf. [6]).

§ 3. Local boundedness.

Let $(\bar{x}, \bar{t})$ be a fixed point in the basic set $Q = \Omega \times (0, T)$. We denote by $R(\rho)$ the open cube in E^n of edge length ρ centered at $\bar{x}$, and define

$$Q(\rho) = R(\rho) \times (\bar{t} - \rho^a, \bar{t}).$$

We denote by $\|\cdot\|_{p,q,\rho}$ the $L^{p,q}$ norm of a function over the $Q(\rho)$. Then we have the following theorem :

Theorem 3. 1. Let u be a weak solution of (2. 1) in Q. Suppose that

$Q(3\rho)$ is contained in Q. Then almost everywhere in $Q(\rho)$

$$\max_{Q(\rho)} u(x, t) \leq C(\|u\|_{a,a,2\rho} + k_0)^m$$

for some constant m depending only on a, n, where C is a constant depending on ρ and the structure (2. 1), and $k_0 = (\|e\| + \|f\|)^{\frac{1}{a-1}} + \|g\|^{\frac{1}{a}}$.

Proof. In the fundamental inequality (2. 9), we set $v = \bar{u}^q$ for $q \geqq 1$, where $\bar{u} = \max(0, u) + k_0$. Then we see

$$(3.\ 1)\qquad \frac{1}{\beta+1}\int \varphi^a\{\bar{u}^{\beta+1} - (\beta+1)k_0^\beta \bar{u} + \beta k_0^{\beta+1}\}_{t=\tau}\, dx + a\beta q^{-a}\iint \varphi^a |v_x|^a\, dxdt$$

$$\leqq \iint [(a\bar{a}|\varphi_x|\varphi^{a-1} + b\varphi^a)q^{1-a} v|v_x|^{a-1} + G_1(x, t)v^a]\, dxdt$$

$$+ \frac{a}{\beta+1}\iint \varphi^{a-1}|\varphi_t|\bar{u}^{\beta+1} dxdt.$$

Using Lemmas 2. 1, 2. 2 and 2. 3, we shall estimate each term of the inequality (3. 1).

First we assume that $a < n$. By Young's inequality

$$(3.\ 2)\qquad \iint \bar{a}|\varphi_x|\varphi^{a-1} q^{1-a} v|v_x|^{a-1} dxdt \leqq \eta q^{-a}\beta\|\varphi v_x\|_{a,a}^a + c_0\beta^{1-a}\|\varphi_x v\|_{a,a}^a$$

for arbitrary $\eta > 0$, where c_0 is a constant depending only on $\bar{a}$, η and a, which we denote by $c_0(\bar{a}, \eta, a)$.

Using Lemma 2. 3, we see

$$q^{1-a}\int_\Omega b\varphi^a v|v_x|^{a-1} dx = q^{1-a}\int_\Omega (b' + b'')(\varphi v)|\varphi v_x|^{a-1} dx$$

$$\leqq q^{1-a}\max|b'|\int \varphi v|\varphi v_x|^{a-1} dx + q^{1-a}\int b''(\varphi v)|\varphi v_x|^{a-1} dx$$

$$\leqq q^{1-a} B_{\eta'}\|\varphi v\|_a(t)\cdot\|\varphi v_x\|_a^{a-1}(t) + q^{1-a}\|b''\|_n(t)\cdot\|\varphi v_x\|_a^{a-1}(t)\cdot\|\varphi v\|_{a^*}(t)$$

$$\leq c_1 B^a_{\eta'} (\|\varphi v\|^a_a(t) + \beta q^{-a}\eta\|\varphi v_x\|^a_a(t) + q^{1-a}\eta'\|\varphi v_x\|^{a-1}_a(t)\cdot S(\|\varphi_x v\|_a(t)$$

$$+\|\varphi v_x\|_a(t)).$$

Here we put $\eta' = S^{-1}\beta\eta q^{-1}$. Then we may take $B_{\eta'} = C(M)\times(S^{-1}\beta\eta q^{-1})^{-\frac{n}{\varepsilon}}$.

Using Young's inequality again, we see

$$\leq c_2(\beta^{-1}q)^{\frac{an}{\varepsilon}}\|\varphi v\|^a_a(t) + 3\eta\beta q^{-a}\|\varphi v_x\|^a_a(t) + \eta\beta q^{-a}\|\varphi_x v\|^a_a(t),$$

where $c_2 = c_2(\eta, a, n, \varepsilon, S, M)$.

Integrating by t, we have

$$(3.\ 3) \qquad q^{1-a}\iint b\varphi^a v|v_x|^{a-1}dxdt \leq c_2(\beta^{-1}q)^{\frac{an}{\varepsilon}}\|\varphi v\|^a_{a,a} + 3\eta\beta q^{-a}\|\varphi v_x\|^a_{a,a}$$

$$+ \eta\beta q^{-a}\|\varphi_x v\|^a_{a,a}.$$

Next we consider

$$\int(c + k_0^{1-a}e)|\varphi_x|\varphi^{a-1}v^a dx \leq \|c + k_0^{1-a}e\|_{\frac{n}{a-1}}(t)\cdot\|\varphi_x v\|_a(t)\cdot\|\varphi v\|^{a-1}_{a^*}(t)$$

$$\leq (M+1)\|\varphi_x v\|_a(t)\cdot S^{a-1}(\|\varphi_x v\|_a(t) + \|\varphi v_x\|_a(t))^{a-1}$$

$$\leq c_4 q^{a(a-1)}\beta^{1-a}\|\varphi_x v\|^a_a(t) + \eta\beta q^{-a}\|\varphi v_x\|^a_a(t),$$

where $c_4 = c_4(\eta, a, n, S, M)$.

Thus we have

$$(3.\ 4) \qquad \iint(c + k_0^{1-a}e)|\varphi_x|\varphi^{a-1}v^a dxdt \leq c_4 q^{a(a-1)}\beta^{1-a}\|\varphi_x v\|^a_{a,a} + \eta\beta q^{-a}\|\varphi v_x\|^a_{a,a}.$$

Next clearly

$$\int\{(\beta+1)d + k_0^{1-a}f + \beta k_0^{-a}g\}\varphi^a v^a dx = (\beta+1)\int(d + k_0^{1-a}f + k_0^{-a}g)\varphi^a v^a dx$$

$$\leqq (\beta+1)D_{\eta'}\|\varphi v\|_{\alpha}^{\alpha}(t) + (\beta+1)\eta'\|\varphi v\|_{\alpha *}^{\alpha}(t)$$

$$\leqq (\beta+1)D_{\eta'}\|\varphi v\|_{\alpha}^{\alpha}(t) + 2^{\alpha} S^{\alpha}(\beta+1)\eta'(\|\varphi_x v\|_{\alpha}^{\alpha}(t) + \|\varphi v_x\|_{\alpha}^{\alpha}(t)).$$

Here we put $\eta' = (2^{\alpha} S^{\alpha}(\beta+1))^{-1}\beta q^{-\alpha}\eta$ for any $\eta > 0$. Then we may take $D_{\eta'} = c(M)(2^{\alpha} S^{\alpha}(\beta+1))^{\frac{n}{\alpha\varepsilon}}\beta^{-\frac{n}{\alpha\varepsilon}} q^{\frac{n}{\varepsilon}}\eta^{-\frac{n}{\varepsilon\alpha}}$. Thus we get

$$\text{(3. 5)} \quad \iint \{(\beta+1)d + k_0^{1-\alpha} f + \beta k_0^{-\alpha} g\}\varphi^{\alpha} v^{\alpha}\, dxdt \leqq c_5\{(\beta+1)\beta^{-1}\}^{\frac{n}{\alpha\varepsilon}} q^{\frac{n}{\varepsilon}}\|\varphi v\|_{\alpha,\alpha}^{\alpha}$$
$$+ \eta\beta q^{-\alpha}\|\varphi v_x\|_{\alpha,\alpha}^{\alpha} + \eta\beta q^{-\alpha}\|\varphi_x v\|_{\alpha,\alpha}^{\alpha},$$

where $c_5 = c_5(\eta, \alpha, n, \varepsilon, S, M)$.

Finally, by Young's inequality, we see

$$\text{(3. 6)} \quad \bar{u}^{\beta+1} - (\beta+1)k_0^{\beta}\bar{u} + \beta k_0^{\beta+1} \geqq \frac{1}{2}\bar{u}^{\beta+1} - \beta k_0^{\beta+1}.$$

From (3. 1) (3. 6) we obtain

$$\frac{1}{2(\beta+1)}\int \varphi^{\alpha}\bar{u}^{\beta+1}]_{t=\tau}\,dx + \beta q^{-\alpha}(a - 5\eta)\iint|\varphi v_x|^{\alpha}\,dxdt$$
$$\leqq c_7 q^{p}\iint v^{\alpha}(\varphi^{\alpha} + |\varphi_x|^{\alpha})\,dxdt + \frac{a}{\beta+1}\iint \varphi^{\alpha-1}|\varphi_t|\bar{u}^{\beta+1}\,dxdt + \frac{\beta}{\beta+1}\int \varphi^{\alpha} k_0^{\beta+1}\,dx$$

where $p = \max(\frac{\alpha n}{\varepsilon} - 1, (\alpha-1)^2, \frac{n}{\varepsilon} + 1)$.

Therefore, putting $\eta = \frac{a}{14}$ and noting the fact that

$$\int \varphi^{\alpha} k_0^{\beta+1}\,dx \leq \frac{1}{\tau}\iint \varphi^{\alpha} k_0^{\beta+1}\,dxdt \leq \frac{1}{\tau}\iint \varphi^{\alpha}\bar{u}^{\beta+1}\,dxdt, \text{ we have}$$

$$\text{(3. 7)} \quad \frac{1}{2\alpha q}\int_{\Omega}\varphi^{\alpha}\bar{u}^{\beta+1}dx]_{t=\tau}\,dx + \frac{a}{2}q^{1-\alpha}\iint|\varphi v_x|^{\alpha}\,dxdt$$
$$\leqq c_3 q^{p}(\iint v^{\alpha}(\varphi^{\alpha} + |\varphi_x|^{\alpha})\,dxdt + \iint(\tau^{-1}\varphi^{\alpha} + |\varphi_t|\varphi^{\alpha-1})\bar{u}^{\beta+1}\,dxdt)$$

for any $\tau \in (0, T)$, where $c_8 = c_8(\eta, \alpha, n, \varepsilon, S, M, a, \bar{a})$.

Let $\rho_\nu = \rho(1+2^{-\nu})$, $\tau_\nu = \rho_\nu^a$ and denote

$$Q_\nu = Q(\rho_\nu) = R(\rho_\nu) \quad (\bar{t} - \tau_\nu, \bar{t}).$$

We choose the function $\varphi_\nu(x, t)$ such that $\varphi_\nu(x, t) = \psi_1^{(\nu)}(x) \cdot \psi_2^{(\nu)}(t)$, where

$$\psi_1^{(\nu)}(x) = \begin{cases} 0 & \text{if } x - \bar{x} \geq \rho_\nu, \\ \dfrac{\rho_\nu - (x - \bar{x})}{\rho_\nu - \rho_{\nu+1}} & \text{if } \rho_\nu > x - \bar{x} \geq \rho_{\nu+1}, \\ 1 & \text{if } x - \bar{x} < \rho_{\nu+1}, \end{cases}$$

and

$$\psi_2^{(\nu)}(t) = \begin{cases} 0 & \text{if } \bar{t} - t \geq \tau_\nu, \\ \dfrac{\tau_\nu - (\bar{t} - t)}{\tau_\nu - \tau_{\nu+1}} & \text{if } \tau_\nu > \bar{t} - t \geq \tau_{\nu+1}, \\ 1 & \text{if } \tau_{\nu+1} > \bar{t} - t \geq 0, \end{cases}$$

Then, noting that

$$\varphi_x{}^a \leqq (\rho_\nu - \rho_{\nu+1})^{-a}, \quad \varphi_t \leqq (\tau_\nu - \tau_{\nu+1})^{-a} \leqq (\rho_\nu - \rho_{\nu+1})^{-a} \quad \text{and}$$

$$\tau^{-1} \leqq (\rho_\nu - \rho_{\nu+1})^{-a},$$

we get

$$(3.\ 8) \begin{cases} \displaystyle\iint_{Q_\nu} |\varphi_\nu v_x|^a dxdt \leqq C_0 q^{p+a-1}(\rho_\nu - \rho_{\nu+1})^{-a} \Big\{\iint_{Q_\nu} v^a dxdt + \Big(\iint_{Q_\nu} v^a dxdt\Big)^{\frac{\beta+1}{aq}}\Big\}, \\ \displaystyle\max_t \int_{R(\rho_\nu)} \varphi_\nu^a \bar{u}^{\beta+1} dx \leqq C_1 q^{p+1}(\rho_\nu - \rho_{\nu+1})^{-a}\Big\{\iint_{Q_\nu} v^a dxdt + \Big(\iint_{Q_\nu} v^a dxdt\Big)^{\frac{\beta+1}{aq}}\Big\}, \end{cases}$$

where C_1 and C_2 are constants depending only on η, a, n, ε, S, M, $\bar{a}$ and a.

Using the Hölder inequality and Lemma 2.1, we see from (3. 8),

$$(3.\ 9)\qquad \iint_{Q_{\nu+1}} \bar{u}^{aq\{1+\frac{1}{n}(a-\frac{a-2}{q})\}}\,dxdt \leqq \iint_{Q_\nu} (\bar{u}^{aq}\varphi_\nu^a)(\bar{u}^{\beta+1}\varphi_\nu^a)^{\frac{a}{n}}\,dxdt$$

$$\leqq \int\{\int|\varphi_\nu v|^{a^*}dx\}^{\frac{a}{a^*}}\{\int\varphi_\nu^a\bar{u}^{\beta+1}dx\}^{\frac{a}{n}}dt$$

$$\leqq S^a(\max_t \int\varphi_\nu^a\bar{u}^{\beta+1}dx)^{\frac{a}{n}}\iint_{Q_\nu}|\varphi_{\nu_x}v+\varphi_\nu v_x|^a dxdt$$

$$\leqq \begin{cases}[C_2 q^{p+a}(\rho_\nu-\rho_{\nu+1})^{-a}(\iint_{Q_\nu} v^a\,dxdt)^{\frac{\beta+1}{aq}}]^{(1+\frac{a}{n})} & \text{if } \iint_{Q_\nu} v^a dxdt<1,\\ [C_2 q^{p+a}(\rho_\nu-\rho_{\nu+1})^{-a}\iint_{Q_\nu} v^a\,dxdt]^{(1+\frac{a}{n})} & \text{if } \iint_{Q_\nu} v^a dxdt\geq 1.\end{cases}$$

Put $q_\nu=(\frac{n+a}{n})^\nu\cdot\frac{2}{a}+(1-\frac{2}{a})$ and $\gamma_\nu=1+\frac{1}{n}(a-\frac{a-2}{q_{\nu-1}})$. Then $q_0=1$, and $q_\nu=q_{\nu-1}\gamma_\nu=q_{\nu-2}\gamma_{\nu-1}\gamma_\nu=\cdots=\prod_{j=1}^{\nu}\gamma_j$.

Define

$$\phi_\nu=(\rho_\nu^{-(n+a)}\iint_{Q_\nu}\bar{u}^{aq_\nu}dxdt)^{\frac{1}{aq_\nu}}$$

Then from (3. 9),

$$\phi_{\nu+1}^{aq_{\nu+1}}\leqq\begin{cases}[C_2 q_\nu^{p+a}(\rho_\nu/(\rho_\nu-\rho_{\nu+1}))^a(\rho_\nu/\rho_{\nu+1})^n\phi_\nu^{\beta_{\nu+1}}]^{(1+\frac{a}{n})} & \text{if } \phi_\nu<1,\\ [C_2 q_\nu^{p+a}(\rho_\nu/(\rho_\nu-\rho_{\nu+1}))^a(\rho_\nu/\rho_{\nu+1})^n\phi_\nu^{aq_\nu}]^{(1+\frac{a}{n})} & \text{if } \phi_\nu\geqq 1,\end{cases}$$

where $\beta_\nu=aq_\nu-a+1$. From the above we see immediately

$$\phi_{\nu+1}\leqq\{[C_2 q_\nu^{p+a}(\rho_\nu/(\rho_\nu-\rho_{\nu+1}))^a(\rho_\nu/\rho_{\nu+1})^n]^{\frac{1}{aq_{\nu+1}}}\phi_\nu^{s_\nu}\}^{(1+\frac{a}{n})},$$

where $s_\nu=\dfrac{\beta_{\nu+1}}{aq_{\nu+1}}$ or $\dfrac{1}{\gamma_{\nu+1}}$.

By iteration we have

$$(3.\ 10)\qquad \phi_{\nu+1}\leqq[C_2^{\sum_{j=0}^{\nu}\frac{1}{aq_{j+1}}}\prod_{j=0}^{\nu}q_j^{(p+a)\frac{1}{aq_{j+1}}}\prod_{j=0}^{\nu}(\frac{\rho_j}{\rho_j-\rho_{j+1}})^{1/q_{j+1}}(\frac{\rho_j}{\rho_{j+1}})^{\frac{n}{a}\frac{1}{q_{j+1}}}\phi_0]^{\prod_{j=0}^{\nu}(1+\frac{a}{n})s_\nu}.$$

Since $(1+\frac{2}{n})^j \leqq q_j \leqq (1+\frac{a}{n})^j$ for $j = 1, 2, \cdots$, we see $\sum_{j=0}^{\nu} \frac{1}{aq_{j+1}} \leqq \frac{1}{a}\sum_{j=0}^{\nu} (1+\frac{2}{n})^{-(j+1)} \leqq \frac{n}{2a}$. Hence $C_2^{\sum_{j=0}^{\nu}\frac{1}{q_{j+1}}} \leqq C_2^{\frac{n}{2a}}$, $\prod_{j=0}^{\nu} q_j^{(p+a)\frac{1}{aq_{j+1}}} \leqq (1+\frac{a}{n})^{\frac{p+a}{a}\sum j(1+\frac{2}{n})^{-(j+1)}} \leqq (1+\frac{a}{n})^{\frac{p+a}{a}(\frac{n}{2})^2}$,

and

$$\prod_{j=0}^{\nu} \left(\frac{\rho_j}{\rho_j - \rho_{j+1}}\right)^{\frac{1}{q_{j+1}}} \leqq 2^{\sum_{j=0}^{\nu}(j+2)(\frac{n}{n+2})^{j+1}} < 2^{2(\frac{n}{n+2})^2(\frac{n}{2})^2},$$

$$\left(\frac{\rho_j}{\rho_{j+1}}\right)^{\frac{n}{a}\Sigma(\frac{n}{n+2})^{j+1}} \leqq 2^{\frac{n^3}{2a(n+2)}}.$$

We note that

$$1 \leqq \frac{(1+\frac{a}{n})}{\gamma_{j+1}} = \frac{1+\frac{a}{n}}{1+\frac{a}{n}(1-\frac{a-2}{aq_j})} = \left(1-\frac{a-2}{(n+a)q_j}\right)^{-1} \quad \text{and} \quad \frac{a-2}{(n+a)q_j} \leqq \frac{1}{2}.$$

Thus we have

$$\prod_{j=0}^{\nu} (1+\frac{a}{n})s_j \leqq \prod \left\{1+\frac{2(a-2)}{(n+a)q_j}\right\} \leqq \exp\left\{\frac{2(a-2)}{n+a}\Sigma q_j^{-1}\right\} \leqq e^{a-2},$$

that is, $m = (1+\frac{a}{n})s_j \leq e^{a-2}$.

Consequently we can let ν tend to infinity in (3. 12) to obtain

$$\max_{Q(\rho)} \bar{u} \leq C\|\bar{u}\|^{m}_{a,a,2\rho},$$

from which we have Theorem 3.1 for $a < n$, since a similar result can be obtained for $-u$.

Next we assume that $a \geq n$. Using Lemmas 2. 2 and 2. 3, we shall estimate each term of (3. 1).

Consider

$$q^{1-a}\int b\varphi^a v|v_x|^{a-1}dx = q^{1-a}\int(b' + b'')\varphi^a v|v_x|^{a-1}dx$$

$$\leq q^{1-a}\sup|b'|\,\|\varphi v\|_a(t)\cdot\|\varphi v_x\|_a^{a-1}(t)$$

$$+ q^{1-a}\|b''\|_{a(1+\varepsilon)}\|\varphi v\|_{\frac{a(1+\varepsilon)}{\varepsilon}}(t)\cdot\|\varphi v_x\|_a^{a-1}(t)$$

$$\leq c_1\beta^{1-a}B^a_{\eta'}\|\varphi v\|_a^a(t) + \eta\beta q^a\|\varphi v_x\|_a^a(t) + q^{1-a}\eta' S_1(\|\varphi_x v\|_a(t)$$

$$+\|\varphi v_x\|_a(t))\cdot\|\varphi v_x\|_a^{a-1}(t),$$

where $S_1 = S|\Omega|^{\frac{a(1+\varepsilon)-n}{na(1+\varepsilon)}}$. Here we put $\eta' = S_1^{-1}q^{-1}\beta\eta$. Then we may put $B_{\eta'} = c(M)(S_1^{-1}q^{-1}\beta\eta)^{-\frac{1+\varepsilon}{\varepsilon}}$. Thus using Young's inequality again and integrating by t, we get

$$(3.\ 3)' \quad q^{1-a}\iint b\varphi^a v|v_x|^{a-1}dxdt \leq c_2\beta^{1-a-\frac{a(1+\varepsilon)}{\varepsilon}}q^{\frac{a(1+\varepsilon)}{\varepsilon}}\|\varphi v\|^a_{a,a}$$

$$+ 3\eta\beta q^{-a}\|\varphi v_x\|^a_{a,a} + \eta\beta q^{-a}\|\varphi_x v\|^a_{a,a}.$$

Next we can get

$$a\int(c + k_0^{1-a}e)|\varphi_x|\varphi^{a-1}v^a\,dx \leq a\|c + k_0^{1-a}e\|_{\frac{a}{a-1}(1+\varepsilon)}\cdot\|\varphi_x v\|_a(t)\cdot\|\varphi v\|^{a-1}_{\frac{a(1+\varepsilon)}{\varepsilon}}(t)$$

$$\leq a(M+1)\|\varphi_x v\|_a(t)\cdot 2^{a-1}S_1^{a-1}(\|\varphi_x v\|_a^{a-1}(t) + \|\varphi v_x\|_a^{a-1}(t))$$

$$\leq c_3(1 + \beta^{1-a}q^{a(a-1)})\|\varphi_x v\|_a^a(t) + \eta\beta q^{-a}\|\varphi v_x\|_a^a(t).$$

Therefore we obtain

$$(3.\ 4)' \quad a\iint(c + k_0^{1-a}e)|\varphi_x|\varphi^{a-1}v^a\,dxdt \leq c_3(1+\beta^{1-a}q^{a(a-1)})\|\varphi_x v\|^a_{a,a}$$

$$+ \eta\beta q^{-a}\|\varphi v_x\|^a_{a,a}.$$

Similarly

$$\int \{(\beta+1)d + k_0^{1-a} f + \beta k_0^{-a} g\} \varphi^a v^a dx \leqq (\beta+1)\int (d + k_0^{1-a} f + k^{-a} g) \varphi^a v^a dx$$

$$\leqq (\beta+1)D_{\eta'} \|\varphi v\|_a^a (t) + (\beta+1)\eta' \|\varphi v\|^a_{\frac{a(1+\varepsilon)}{\varepsilon}}(t)$$

$$\leqq (\beta+1)D_{\eta'} \|\varphi v\|_a^a (t) + (\beta+1)\eta' 2^a S_1^a (\|\varphi_x v\|_a^a (t) + \|\varphi v_x\|_a^a (t)).$$

Now we put $\eta' = (\beta+1)^{-1} 2^{-a} S_1^{-a} \beta q^{-a} \eta$. Then we may take $D_{\eta'} = c(M)\{(\beta+1)^{-1} 2^{-a} S_1^{-a} \beta q^{-a} \eta\}^{-\frac{1+\varepsilon}{\varepsilon}}$. Thus we see

$$(3.\ 5)' \quad \iint \{(\beta+1)d + k_0^{1-a} f + \beta k_0^{-a} g\} \varphi^a v^a dxdt$$
$$\leqq c_4(\beta+1)^{\frac{1+\varepsilon}{\varepsilon}+1} \beta^{-\frac{1+\varepsilon}{\varepsilon}} q^{a(\frac{1+\varepsilon}{\varepsilon})} \|\varphi v\|^a_{a,a} + \eta\beta q^{-a} \|\varphi_x v\|^a_{a,a} + \eta\beta q^{-a} \|\varphi v_x\|^a_{a,a}$$

From (3. 2), (3. 3)', (3. 4)' and (3. 5)' we obtain (3. 8), where $p = \frac{a(1+\varepsilon)}{\varepsilon} + 1$.

Using Lemma 2.2 and (3. 8), we obtain

$$\iint_{Q_{\nu+1}} \bar{u}^{aq\{1+\frac{1}{a(1+\varepsilon)}(a-\frac{a-2}{q})\}} dxdt \leqq \iint_{Q_\nu} (v\varphi_\nu)^a (\bar{u}^{\beta+1} \varphi_\nu^a)^{\frac{1}{1+\varepsilon}} dxdt$$
$$\leqq \int (\int |v\varphi_\nu|^{\frac{a(1+\varepsilon)}{\varepsilon}} dx)^{\frac{\varepsilon}{1+\varepsilon}} (\int \bar{u}^{\beta+1} \varphi_\nu^a dx)^{\frac{1}{1+\varepsilon}} dt$$
$$\leqq S_1^a (\max_t \int \bar{u}^{\beta+1} \varphi_\nu^a dx)^{\frac{1}{1+\varepsilon}} (\iint |\varphi_{\nu_x} v + \varphi_\nu v_x|^a dxdt)$$
$$\leqq \begin{cases} [C_2 q^{p+a} (\rho_\nu - \rho_{\nu+1})^{-a} (\iint_{Q_\nu} v^a dxdt)^{\frac{\beta+1}{aq}}]^{(1+\frac{1}{1+\varepsilon})} & \text{if } \iint_{Q_\nu} v^a dxdt < 1, \\ [C_2 q^{p+a} (\rho_\nu - \rho_{\nu+1})^{-a} (\iint_{Q_\nu} v^a dxdt)]^{1+\frac{1}{1+\varepsilon}} & \text{if } \iint_{Q_\nu} v^a dxdt \geqq 1. \end{cases}$$

Putting $q_\nu = (1+\frac{1}{1+\varepsilon})^\nu (\frac{2}{a}) + \frac{a-2}{a}$, by same methods as before, we have Theorem 3.1 when $n \leqq a$.

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§ 4. Hölder continuity.

Moser [3] proved the Hölder continuity of solutions for linear parabolic equations from the Harnack inequality. In this section we shall derive the Holder continuity in the same manner as in [3] by using the following Lemma.

Main Lemma. Let u be a non-negative weak solution of (2. 1) in Q, and let $Q(3\rho)$ is contained in Q. Then there exists a constant m depending only on n and α such that

$$(4.1)\qquad \max_{Q(\rho)} u \le C(\rho^{-(n+\alpha)} \| u + k\rho^{\theta} \|_{\alpha,\alpha,\,Q(2\rho)})^{m} (\min_{Q(\rho)} u + k\rho^{\theta}) \quad \text{for} \quad \rho < 1,$$

where C is a constant depending only on the structure of (2. 1), θ is a sufficiently small constant with $0 < \theta < 1$ and $k = (e + f)^{\frac{1}{\alpha-1}} + g^{\frac{1}{\alpha}} + 1$. Here max and min mean the essential maximum and the essential minimum respectively.

We shall prove Main Lemma in the next section.

Using Main Lemma and the following Lemma 4.1, we immediately obtain the following:

Theorem 4. 1. Let u be a weak solution of (2. 1) in Q. Then u is (essentially) Hölder continuous in Q.

Lemma 4. 1. Suppose that for $0 < \eta < 1$ and $0 < \theta < 1$,

$$(4.2)\qquad \omega(\rho) \le \eta(\omega(3\rho) + k\rho^{\theta}),$$

where $\omega(\rho)$ means the oscillation of u on $Q(\rho)$ and k is a constant. Then u is Holder continuous.

Proof. We define constants λ and δ by means of

$$\eta = 3^{-\lambda}, \quad \delta = \mathrm{Min}\,(\lambda,\ \theta/2).$$

Then

$$\omega(\rho) \leq 3^{-\delta}\{\omega(3\rho) + k\rho^{\theta}\}$$

and by iteration

$$\omega(3^{-\nu}\rho) \leq 3^{-\delta\nu}\{\omega(3\rho) + [1 + 3^{\delta-\theta} + \cdots + 3^{\nu(\delta-\theta)}]k\rho^{\theta}\}$$

$$\leq 3^{-\delta\nu}\{\omega(3\rho) + (1 - 3^{-\theta/2})^{-1}k\rho^{\theta}\}.$$

Since the oscillation of u on $Q(\rho)$ tends to zero as $Q(\rho)$ shrinks, it follows that u is Hölder continuous.

Proof of Theorem 4. 1. Since solutions of (2. 1) are locally bounded functions by Theorem 3.1, we may assume that $u < L$ in Q. Let $M(\rho)$ and $m(\rho)$ be respectively the essential maximum and the essential minimum of u in $Q(\rho)$. Let w denote either $M(3\rho) - u$ or $u - m(3\rho)$. Then since $M(3\rho) < L$ and $m(3\rho) < L$, it is easily verified that w is a non-negative weak solution of the equation

$$w_t - \mathrm{div}\,\overline{\mathcal{A}}(x,\ t,\ w,\ w_x) + \overline{\mathcal{B}}(x,\ t,\ w,\ w_x) = 0,$$

where

$$\overline{\mathcal{A}}(x,\ t,\ w,\ p) \leq \overline{a}|p|^{\alpha-1} + \overline{c}|w|^{\alpha-1} + \overline{e}, \quad \overline{c} = 2^{\alpha-1}c, \quad \overline{e} = \overline{c}L^{\alpha-1} + e$$

and so forth. Thus we may apply Main Lemma to w and we obtain

$$M(3\rho) - m(\rho) \leq C'(M(3\rho) - M(\rho) + \overline{k}\rho^{\theta}),$$

$$M(\rho) - m(3\rho) \leq C'(m(\rho) - m(3\rho) + \bar{k}\rho^{\theta}).$$

Here $C' = C(2L + \bar{k})^m$ and $\bar{k} = (\|\bar{e}\| + \|\bar{f}\|)^{\frac{1}{a-1}} + \|\bar{g}\|^{\frac{1}{a}} + 1$. Adding two preceding inequalities, we have

$$\omega(3\rho) + \omega(\rho) \leq C'\{\omega(3\rho) - \omega(\rho) + 2\rho^{\theta}\bar{k}\}.$$

Thus

$$\omega(\rho) \leq \frac{C' - 1}{C' + 1}\{\omega(3\rho) + \frac{2C'}{C' - 1}\bar{k}\rho^{\theta}\}$$

from which Lemma 4.1 implies that u is Hölder continuous.

Remark. We can choose C' to be independent of ρ for $\rho < 1$.

§ 5. Proof of Main Lemma.

In this section, we consider a non-negative weak solution u of (2. 1) in Q. First, we take two lemmas which are needed to prove Main Lemma.

Lemma 5. 1. There exists a constant m_0 depending only on a and n such that

$$\rho^{m_0(n+a)/a} C_0 \|\bar{u}\|^{-m_0}_{a,a,Q(2\rho)} \leq \min_{Q(\rho)} \bar{u}, \tag{5. 1}$$

where $\bar{u} = u + k\rho^{\theta}$ for $0 < \theta < \frac{\varepsilon a}{n + \varepsilon a}$, and C_0 is a constant depending only on the structure of (2. 1). Here it is assumed that $\rho < 1$.

Proof. In the fundamental estimate (2. 10) we put $v = \bar{u}^{q}$ $(q = 0)$. Put $k_1 = k\rho^{\theta}$. Then

$$\frac{1}{|\beta+1|}\int \varphi^{a}\bar{u}^{\beta+1}]_{t=\tau}dx + a/\beta q^{-a}/\iint \varphi^{a}/v_x|^{a}\,dxdt \tag{5. 2}$$

$$\leqq \iint [(a\bar{a}|\varphi_x|\varphi^{a-1} + b\varphi^a)|q|^{1-a} \ v|v_x|^{a-1}G_2(x, t)v^a + \frac{a}{|\beta+1|}\varphi^{a-1}|\varphi_t|k_1^{2-a} v^a] dxdt,$$

where

$$G_2(x, t) = ac|\varphi_x|\varphi^{a-1} + (1+|\beta|)d\varphi^a + ak_1^{1-a} e|\varphi_x|\varphi^{a-1} + k_1^{1-a} f\varphi^a + |\beta|k_1^{-a} g\varphi^a .$$

We estimate each term of (5. 2) by using Lemmas 2.1, 2.2 and 2.3..

(I) We suppose that $a < n$. Using Young's inequality, we have

$$(5.\ 3) \qquad |q|^{1-a} \iint a\bar{a}|\varphi_x|\varphi^{a-1}v|v_x|^{a-1}dxdt \leqq \eta|\beta q^{-a}| \, \|\varphi v_x\|_{a,a}^a + C_0|\beta|^{1-a} \|\varphi_x v\|_{a,a}^a ,$$

where $C_0 = C_0(a, \eta, \bar{a})$.

Now it is seen that

$$|q^{1-a}| \int b\varphi^a v|v_x|^{a-1}dx \leqq |q^{1-a}| B_{\eta'} \|\varphi v\|_a(t) \cdot \|\varphi v_x\|_a^{a-1}(t) +$$

$$+ |q^{1-a}| \eta' \|\varphi v\|_{a*}(t) \cdot \|\varphi v_x\|_a^{a-1}(t)$$

$$\leqq C_1|\beta|^{1-a} B_{\eta'}^a \|\varphi v\|_a^a(t) + \eta|\beta q^{-a}| \, \|\varphi v\|_a^a(t) + S|q|^{1-a} \ \eta' (\|\varphi_x v\|_a(t)$$

$$+ \|\varphi v_x\|_a(t)) \times \|\varphi v_x\|_a^{a-1}(t).$$

Here, we put $\eta' = S^{-1}|\beta q^{-1}|$. Then, we may put $B_{\eta'} = c(M)(S^{-1}|\beta q^{-1}|)^{-\frac{n}{\varepsilon}}$.

Thus, we have

$$(5.\ 4) \qquad |q|^{1-a} \iint b\varphi^a v|v_x|^{a-1}dxdt \leqq C_2|\beta|^{1-a-\frac{an}{\varepsilon}} |q|^{\frac{an}{\varepsilon}} \|\varphi v\|_{a,a}^a + 3\eta|\beta q^{-a}| \, \|\varphi v_x\|_{a,a}^a$$

$$+ \eta|\beta q^{-a}| \, \|\varphi_x v\|_{a,a}^a .$$

Next

$$\int a(c + k_1^{1-a} e)|\varphi_x|\varphi^{a-1}v^a dx \leqq a\rho^{(1-a)\theta}\int (c + k^{1-a} e)|\varphi_x|\varphi^{a-1}v^a dx$$

$$\leq a\rho^{(1-a)\theta}\|c + k^{1-a} e\|_{\frac{n}{a-1}}(t)\cdot\|\varphi_x v\|_a(t)\cdot\|\varphi v\|_{a^*}^{a-1}(t)$$

$$\leq a\rho^{(1-a)\theta}(M+1)\cdot\|\varphi_x v\|_a(t)\cdot 2^{a-1}S^{a-1}(\|\varphi_x v\|_a^{a-1}(t) + \|\varphi v_x\|_a^{a-1}(t))$$

$$\leq C_3(1+|\beta|^{1-a}|q|^{a(a-1)})\rho^{a(1-a)\theta}\cdot\|\varphi_x v\|_a^a(t) + \eta|\beta q^{-a}|\,\|\varphi v_x\|_a^a(t).$$

Therefore, we get

$$\iint a(c+k_1^{1-a}e)|\varphi_x|\varphi^{a-1}v^a\,dxdt \leq C_3(1+|\beta|^{1-a}|q|^{a(a-1)})\rho^{a(1-a)\theta}\|\varphi_x v\|_{a,a}^a + \eta|\beta q^{-a}|\,\|\varphi v_x\|_{a,a}^a, \tag{5. 5}$$

where $C_3 = C_3(a, \eta, M, S)$.

Finally it is verified that

$$\int\{(1+|\beta|)d + k_1^{1-a}f + |\beta|k_1^{-a}g\}\varphi^a v^a\,dx \leq (1+|\beta|)\rho^{-a\theta}\int(d+k^{1-a}f + k^{-a}g)dx$$

$$\leq (1+|\beta|)\rho^{-a\theta}D_{\eta'}\|\varphi v\|_a^a(t) + (1+|\beta|)\rho^{-a\theta}\eta'\|\varphi v\|_{a^*}^a(t)$$

$$\leq (1+|\beta|)\rho^{-a\theta}D_{\eta'}\|\varphi v\|_a^a(t) + 2^a\cdot S^a(1+|\beta|)\rho^{-a\theta}(\|\varphi_x v\|_a^a(t) + \|\varphi v_x\|_a^a(t)).$$

We put $\eta' = \eta|\beta q^{-a}|(2^a S^a(1+|\beta|)\rho^{-a\theta})^{-1}$. Then we may put $F_{\eta'} = c(M)\,\eta|\beta q|^{-a\,-\frac{n}{a\varepsilon}}(2^a S^a(1+|\beta|)\rho^{-a\theta})^{\frac{n}{a\varepsilon}}$, and we obtain

$$\iint\{(1+|\beta|)d + k_1^{1-a}f + |\beta|k_1^{-a}g\}\varphi^a v^a\,dxdt \tag{5. 6}$$

$$\leq C_4(1+|\beta|)^{1+\frac{n}{a\varepsilon}}|\beta|^{-\frac{n}{a\varepsilon}}|q|^{\frac{n}{\varepsilon}}\rho^{-a\theta(1+\frac{n}{a\varepsilon})}\|\varphi v\|_{a,a}^a + \eta|\beta q^{-a}|\,\|\varphi_x v\|_{a,a}^a$$

$$+ \eta|\beta q^{-a}|\,\|\varphi v_x\|_{a,a}^a.$$

From (5. 2), (5. 3), (5. 4), (5. 5) and (5. 6) we get

(5. 7) $$\frac{1}{|\beta+1|}\int \varphi^{a}\bar{u}^{\beta+1}\Big]_{t=\tau} dx + |\beta q^{-a}|(a-6q)\iint \varphi^{a}|v_x|^{a}\,dxdt$$

$$\leq C_5\{|\beta|^{1-a} + |\beta|^{1-a-\frac{an}{\varepsilon}}|q|^{\frac{an}{\varepsilon}} + |\beta|^{1-a}|q|^{a(a-1)} + (1+|\beta|)^{1+\frac{n}{a\varepsilon}}|\beta|^{-\frac{n}{a\varepsilon}}|q|^{\frac{n}{\varepsilon}} + 1\}$$

$$(\iint(\rho^{-\theta a(1+\frac{n}{\varepsilon a})}\varphi^{a} + \rho^{-\theta a(a-1)}|\varphi_x|^{a} + \rho^{-\theta(a-2)}\varphi^{a-1}|\varphi_t|)v^{a}\,dxdt).$$

Let $q < \frac{a-2}{a}$. Then $\beta < -1$. If $q < -1$, then $\beta < 1-2a$, and $K_{\beta q} = |\beta|^{1-a} + |\beta|^{1-a-\frac{an}{\varepsilon}}|q|^{\frac{an}{\varepsilon}} + |\beta|^{1-a}|q|^{a(a-1)} + (1+|\beta|)^{1+\frac{n}{a\varepsilon}}|\beta|^{-\frac{n}{a\varepsilon}}|q|^{\frac{n}{\varepsilon}} + 1 \leq C q^{\frac{n}{\varepsilon}}$, where c is a constant depending only on a. If $-1 < q < \frac{a-2}{a+n}$, then $K_{\beta q} < C$.

Let $\rho_\nu = \rho(1+2^{-\nu})$, $\tau_\nu = \rho_\nu^{a}$ and $Q_\nu = Q(\rho_\nu) = R(\rho_\nu)\ (\bar{t}-\tau_\nu, \bar{t})$ as before. We choose the function $\varphi_\nu(x, t)$ as in § 3. Then, noting that $\theta a(1+\frac{n}{\varepsilon a}) < a$, $\rho^{-a} \leq (\rho_\nu - \rho_{\nu+1})^{-a}$ and $\rho^{-\theta a(a-1)} > \rho^{-\theta(a-2)} > 1$, we see that the inequality (5. 7) yields

(5. 8) $$\begin{cases} \iint_{Q_\nu}|\varphi_\nu v_x|^{a}\,dxdt \leq K_1(\rho_\nu - \rho_{\nu+1})^{-a}\rho^{-\theta a(a-1)}\iint_{Q_\nu} v^{a}\,dxdt, \\ \max_t \int_{R(\rho_\nu)}\varphi_\nu^{a}\bar{u}^{\beta+1}dx \geq K_2(\rho_\nu - \rho_{\nu+1})^{-\alpha}\rho^{-\theta\alpha(\alpha-1)}\iint_{Q_\nu} v^{\alpha}\,dx\,dt \end{cases}$$

where if $q < -1$, $K_1 = C|q|^{\frac{n}{\varepsilon}+a-1}$, $K_2 = C|q|^{\frac{n}{\varepsilon}+1}$ and if $-1 < q < \frac{a-2}{n+a}$, then $K_1 = K_2 = C$ (C is a constant depending only on the structure of (2. 1)).

Put $q_0 = \frac{(a-2)(n+a)}{2a(2n+a)}$ and define

$$q_\nu = (1+\frac{a}{2n})^{\nu}\{q_0 - (\frac{n+a}{a})(\frac{a-2}{a})\} + (\frac{n+a}{a})(\frac{a-2}{a}).\quad \text{Then}$$

$$q_\nu = q_{\nu-1}\{\frac{1}{2} + \frac{n+a}{2n}(1-\frac{a-2}{aq_{\nu-1}})\} = (1+\frac{a}{2n})q_{\nu-1} - (\frac{n+a}{2n})(\frac{a-2}{a}).$$

Moreover, if we put $\gamma_\nu = \frac{1}{2} + \frac{n+a}{2n}(1-\frac{a-2}{aq_{\nu-1}})$, then $q_\nu = \gamma_\nu q_{\nu-1} = \cdots = q_0\prod_{j=1}^{\nu}\gamma_j$.

Define $\phi_\nu = (\rho_\nu^{-(n+a)}\iint_{Q_\nu}\bar{u}^{aq_\nu}\,dxdt)^{\frac{1}{a|q_\nu|}}$. Then from (5. 8),

$$\phi_{\nu+1}^{a|q_{\nu+1}|} = \rho_{\nu+1}^{-(n+a)} \iint_{Q_{\nu+1}} \bar{u}^{aq_{\nu+1}} dxdt \leqq \rho_{\nu+1}^{-(n+a)} \iint_{Q} (v_\nu \varphi_\nu)^{a/2} (\bar{u}^{\beta_\nu+1} \varphi_\nu^a)^{\frac{n+a}{2n}} dxdt$$

$$\leqq \rho_{\nu+1}^{-(n+a)} \int \{ \int |v_\nu \varphi_\nu|^{a^*} dx \}^{a/2a^*} \{ \int (\bar{u}^{\beta_\nu+1}) dx \}^{\frac{n+a}{2n}} dt$$

$$\leqq \rho_{\nu+1}^{-(n+a)} S^{\frac{a}{2}} (\max \int \bar{u}^{\beta_\nu+1})^{\frac{n+a}{2n}} (\iint |v_\nu \varphi_\nu|^a dx)^{\frac{1}{2}} (\int dx)^{\frac{1}{2}}$$

$$\leqq K_1^{\frac{1}{2}} K_2^{\frac{n+a}{2n}} -(n+a) (\rho_\nu - \rho_{\nu+1})^{-a(1+\frac{a}{2n})} \rho^{-\theta a(a-1)(1+\frac{a}{2n})}$$

$$\rho_\nu^{\frac{n}{2}+(n+a)(1+\frac{a}{2n})} \phi^{|aq|(1+\frac{a}{2n})}$$

where $v_\nu = \bar{u}^{q_\nu}$, $\beta_\nu = aq_\nu - a + 1$, and is a constant depending only on the structure of (2. 1). Hence we obtain

$$\phi_{\nu+1} \leqq \{[K_\nu (\rho_\nu / (\rho_\nu - \rho_{\nu+1}))^a]^{(1+\frac{a}{2n})} (\rho_\nu / \rho_{\nu+1})^{(n+a)} \rho_\nu^{\frac{n-a}{2}} \rho_\nu^{-a(a-1)(1+\frac{a}{2n})}\}^{\frac{1}{a|q_{\nu+1}|}} \times$$

$$\times \phi_\nu^{\frac{1}{|\gamma_{\nu+1}|}(1+\frac{a}{2n})},$$

where

$$K_\nu = \begin{cases} |q_\nu|^{\frac{n}{\varepsilon}+a} \cdot (< (1+\frac{a}{2n})^{(\frac{n}{\varepsilon}+a)\nu}) & \text{if } q_\nu < -1 \\ (1+\frac{a}{2n}) & \text{if } -1 < q_\nu < \frac{(a-2)(n+a)}{2a(2n+a)}. \end{cases}$$

By iteration, we have

$$\phi_{\nu+1} \leqq \prod_{j=0}^{\nu} \{[K_j (\rho_j / (\rho_j - \rho_{j+1}))^a]^{\frac{1}{a|q_{\nu+1}|}(1+\frac{a}{2n})^{\nu-j+1}} (\rho_\nu / \rho_{\nu+1})^{(n+a)\frac{1}{a|q_{\nu+1}|}(1+\frac{a}{2n})^{\nu-j}}$$

$$\times \phi_0^{\prod_{j=0}^{\nu} \frac{1}{\gamma_{j+1}}(1+\frac{a}{2n})}.$$

Here we used the fact that $\rho_\nu^{\frac{n-a}{2}} \rho^{-\theta a(a-1)(1+\frac{a}{2n})} < 2$ for a sufficiently small θ, that is, for $\theta \leqq \frac{n(n-a)}{a(a-1)(2n+a)}$.

Noting that $(1+\frac{a}{2n})^j > q_j > q_0(1+\frac{a}{2n})^j$ for $j = 1, 2, \cdots$, we see

$$\frac{(1+\frac{a}{2n})^{\nu-j+1}}{|q_{\nu+1}|} \leqq \frac{1}{q_0}(1+\frac{a}{2n})^{-j} \quad \text{and} \quad \sum_{j=0}^{\nu}(1+\frac{a}{2n})^{-j} \leqq \frac{2n+a}{2n}.$$

Thus

$$\prod_{j=0}^{\nu} C^{\frac{1}{a|q_{\nu+1}|}(1+\frac{a}{2n})^{\nu-j+1}} \leqq C^{\sum_{j=0}^{\nu}\frac{1}{aq_0}(1+\frac{a}{2n})^{-j}} \leqq C^{\frac{1}{aq_0}\frac{2n+a}{2n}}$$

and

$$\prod_{j=0}^{\nu} K_j^{\frac{1}{a|q_{\nu+1}|}(1+\frac{1}{2n})^{\nu-j+1}} \leqq (1+\frac{a}{2n})^{(\frac{n}{\varepsilon}+a)\sum\frac{1}{aq_0}(1+\frac{a}{2n})^{-j}} \leqq (1+\frac{a}{2n})^{(\frac{n}{\varepsilon}+a)\frac{1}{aq_0}\cdot\frac{2n+a}{2n}}.$$

Since $\frac{\rho_\nu}{\rho_\nu - \rho_{\nu+1}} = 2^{\nu+1}+2$ and $(\frac{\rho_\nu}{\rho_{\nu+1}}) \leqq 2$, we get

$$\prod_{j=0}^{\nu}[(\frac{\rho_j}{\rho_j-\rho_{j+1}})]^{\frac{1}{q_{j+1}}(1+\frac{1}{2n})^{\nu-j+1}} \leqq 2^{\frac{1}{q_0}\sum_{j=0}^{\nu}(j+2)(1+\frac{a}{2n})^{-j}} \leqq 2^{\frac{1}{q_0}\frac{a(2n+a)}{4n^2}},$$

$$\prod_{j=0}^{\nu}(\frac{\rho_\nu}{\rho_{\nu+1}})^{\frac{(n+a)}{a q_{\nu+1}}(1+\frac{a}{2n})^{\nu-j}} \leqq 2^{\frac{1}{aq_0}\sum_{j=0}^{\nu}(1+\frac{a}{2n})^{-j-1}} \leqq 2^{\frac{1}{aq_0}}.$$

Therefore we obtain

$$\phi_{\nu+1} \leqq C_0' \phi_0^{\prod_{j=0}^{\nu}(1+\frac{a}{2n})\frac{1}{\gamma_{j+1}}},$$

where C_0' does not depend on ν.

Here, letting ν tend to infinity, we have

(5. 9) $$\max_{Q(\rho)} \bar{u}^{-1} \leqq C_0' \phi_0^{m_0}, \quad m_0 = \prod_{j=0}(1+\frac{a}{2n})\frac{1}{\gamma_{j+1}} < 1.$$

Since

$$\phi_0^{m_0} = \{(2\rho)^{-(n+a)} \iint_{Q(2\rho)} \bar{u}^{aq_0} dxdt\}^{\frac{m_0}{aq_0}}$$

$$\leqq \{(2\rho)^{-(n+a)} \iint_{Q(2\rho)} \bar{u}^{a} dxdt\}^{\frac{m_0}{a}},$$

we get

$$C_0'^{-1}(2\rho)^{\frac{m_0(n+a)}{a}} \| \bar{u} \|_{a,a,Q(2\rho)}^{-m_0} \leqq \min_{Q(\rho)} \bar{u}.$$

(II) We suppose that $a \geqq n$. Let $\gamma = \frac{na(1+\varepsilon)}{n\varepsilon + a(1+\varepsilon)}$. Then $\frac{a(1+\varepsilon)}{\varepsilon} = \gamma^*$.

First clearly

$$|q|^{1-a} \int b\varphi^a v|v_x|^{a-1}dx \leqq |q|^{1-a} B_{\eta'} \|\varphi v\|_a(t) \cdot \|\varphi v_x\|_a^{a-1}(t)$$

$$+ |q|^{1-a} \eta' \|\varphi v_x\|_a^{a-1}(t) \cdot \|\varphi v\|_{\gamma^*}(t)$$

$$\leqq c_0|\beta|^{1-a} B_{\eta'}^a \|\varphi v\|_a^a(t) + |\beta q^{-a}| \eta \|\varphi v_x\|_a^a(t) + S_1|q|^{1-a} \eta' \|\varphi v_x\|_a^{a-1}(t)$$

$$(\|\varphi_x v\|_a(t) + \|\varphi v_x\|_a(t)). \quad \text{(cf. Lemma 2.1).}$$

Put $\eta' = S_1^{-1}|\beta q^{-1}|\eta$. Then we may put $B_{\eta'} = c(M)(S_1^{-1}|\beta q^{-1}|\eta)$. Thus, we get

$$(5.\ 4)' \quad |q|^{1-a} \iint b\varphi^a v|v_x|^{a-1}dxdt \leqq c_1|\beta|^{1-a-\frac{a(1+\varepsilon)}{\varepsilon}} |q|^{\frac{a(1+\varepsilon)}{\varepsilon}} \|\varphi v\|_{a,a}^a$$

$$+ 3\eta|\beta q^{-a}| \|\varphi v_x\|_{a,a}^a + \eta|\beta q^{-a}| \|\varphi_x v\|_{a,a}^a.$$

Next we have

$$\int a(c+k^{1-a}e)|\varphi_x|\varphi^{a-1}v^a dx \leqq \rho^{\theta(1-a)} a \int (c+k^{1-a}e)|\varphi_x|\varphi^{a-1}v^a dx$$

$$\leqq \rho^{\theta(1-a)} a \|c+k^{1-a}e\|_{\frac{a}{a-1}} \cdot \|\varphi_x v\|_a(t) \cdot \|\varphi v\|_{\gamma^*}^{a-1}(t)$$

$$\leqq \rho^{\theta(1-a)} a(M+1) \|\varphi_x v\|_a(t) \cdot 2^{a-1} \cdot S_1^{a-1}(\|\varphi_x v\|_a^{a-1}(t) + \|\varphi v_x\|_a^{a-1}(t))$$

$$\leqq c_2(1+|\beta|^{1-a}|q|^{a(a-1)})\|\varphi_x v\|_a^a(t) + \eta|\beta q^{-a}|\,\|\varphi v_x\|_a^a(t).$$

Here we used the fact that $\rho^{\theta(1-a)}S_1^{a-1} \leqq S^{a-1}$. Therefore, integrating by t, we obtain

$$(5.\ 5)' \quad \iint a(c+k_1^{1-a}e)|\varphi_x|\varphi^{a-1}v^a\,dxdt \leqq c_2(1+|\beta|^{1-a}|q|^{a(a-1)})\|\varphi_x v\|_{a,a}^a$$

$$+\eta|\beta q^{-a}|\,\|\varphi v_x\|_{a,a}^a.$$

Finally we see

$$\int\{(1+|\beta|)d + k_1^{1-a}f + k_1^{-a}|\beta|g\}\varphi^a v^a\,dx \leqq \rho^{-a\theta}(1+|\beta|)\int(d+k^{1-a}f+k^{-a}g)\varphi^a v^a\,dx$$

$$\leqq \rho^{-a\theta}(1+|\beta|)D_{\eta'}\|\varphi v\|_a^a(t) + \rho^{-a\theta}(1+|\beta|)\eta'\|\varphi v\|_{\gamma*}^a(t)$$

$$\leqq \rho^{-a\theta}(1+|\beta|)D_{\eta'}\|\varphi v\|_a^a(t) + 2^a\rho^{-a\theta}S_1^a(1+|\beta|)\eta'(\|\varphi_x v\|_a^a(t) + \|\varphi v_x\|_a^a(t)).$$

Now we put $\eta' = \{2^a\rho^{-a\theta}S_1^a(1+|\beta|)\}^{-1}\eta|\beta q^{-a}|$. Then we may take $D_{\eta'} = c(M)\{2^a S(1+|\beta|)\}^{\frac{1+\varepsilon}{\varepsilon}}(\eta|\beta q^{-a}|)^{-\frac{1+\varepsilon}{\varepsilon}}$. Thus, we have

$$(5.\ 6)' \quad \iint\{(1+|\beta|)d + k_1^{1-a}f + k_1^{-a}|\beta|g\}\varphi^a v^a\,dxdt$$

$$\leqq c_3\rho^{-a\theta}(1+|\beta|)^{1+\frac{1+\varepsilon}{\varepsilon}}|\beta|^{-\frac{1+\varepsilon}{\varepsilon}}|q|^{\frac{a(1+\varepsilon)}{\varepsilon}}\|\varphi v\|_{a,a}^a$$

$$+\eta|\beta q^{-a}|\,\|\varphi_x v\|_{a,a}^a + \eta|\beta q^{-a}|\,\|\varphi v_x\|_{a,a}^a.$$

From (5. 2), (5. 3), (5. 4)', (5. 5)' and (5. 6)', we obtain

$$(5.\ 7)' \quad \begin{cases} \iint_{Q_\nu}|\varphi_\nu v_x|^a\,dxdt \leqq K_1(\rho_\nu-\rho_{\nu+1})^{-a}\rho^{-\theta a}\iint_{Q_\nu}v^a\,dxdt \\ \max_t\int_{R(\rho_\nu)}\varphi_\nu^a\bar{u}^{\beta+1}dx \leqq K_2(\rho_\nu-\rho_{\nu+1})^{-a}\rho^{-\theta a}\iint_{Q_\nu}v^a\,dxdt \end{cases}$$

Putting $q_\nu = (\frac{3}{2})^\nu (q_0 - \frac{\alpha - 2}{\alpha}) + \frac{\alpha - 2}{\alpha}$, we have (5. 1) by the same methods as before, where $q_0 = \frac{\alpha - 2}{6\alpha}$.

Remark. The constant C_0 in (5. 1) does not depend on ρ (< 1).

Lemma 5. 2. There exists a constant m_1 depending only on α and n such that

$$\max_{Q(\rho)} u \leqq \rho^{-m_1(n+\alpha)/\alpha} C_1 \|\bar{u}\|^{m_1}_{\alpha,\alpha,Q(2\rho)}, \tag{5. 10}$$

where $\bar{u} = u + k\rho^\theta$ and C_1 is a constant depending only on the structure of (2. 1).

Since the proof is quite similar to that of Theorem 3.1, we may omit it.

Proof of Main Lemma. Combining the inequalities (5. 1) and (5. 10), we see

$$\max_{Q(\rho)} u \leqq \rho^{-\frac{(n+\alpha)}{\alpha}(m_0+m_1)} C_0^{-1} C_1 \|\bar{u}\|^{m_0+m_1}_{\alpha,\alpha,Q(2\rho)} \min_{Q(\rho)} \bar{u}.$$

Here, putting $m = m_0 + m_1$, we have the lemma.

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